

Perfect Packings in Quasirandom Hypergraphs II

John Lenz *

University of Illinois at Chicago
lenz@math.uic.edu

Dhruv Mubayi †

University of Illinois at Chicago
mubayi@math.uic.edu

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Abstract

For each of the notions of hypergraph quasirandomness that have been studied, we identify a large class of hypergraphs F so that every quasirandom hypergraph H admits a perfect F -packing. An informal statement of a special case of our general result for 3-uniform hypergraphs is as follows. Fix an integer $r \geq 4$ and $0 < p < 1$. Suppose that H is an n -vertex triple system with $r|n$ and the following two properties:

- for every graph G with $V(G) = V(H)$, at least p proportion of the triangles in G are also edges of H ,
- for every vertex x of H , the link graph of x is a quasirandom graph with density at least p .

Then H has a perfect $K_r^{(3)}$ -packing. Moreover, we show that neither hypotheses above can be weakened, so in this sense our result is tight. A similar conclusion for this special case can be proved by Keevash's hypergraph blowup lemma, with a slightly stronger hypothesis on H .

1 Introduction

A k -uniform hypergraph H (k -graph for short) is a collection of k -element subsets (edges) of a vertex set $V(H)$. For a k -graph H and a subset S of vertices of size at most $k - 1$, define the $(k - |S|)$ -graph $N_H(S) := \{T \subseteq V(H) - S : T \cup S \in H\}$. Also, let $d_H(S) = |N_H(S)|$. When $S = \{x\}$, we write $N_H(x)$ and $d_H(x)$. The *minimum ℓ -degree* of H , written $\delta_\ell(H)$, is the minimum of $d_H(S)$ taken over all ℓ -sets $S \in \binom{V(H)}{\ell}$. The *minimum codegree* of H is $\delta_{k-1}(H)$ and the *minimum degree* is $\delta(H) = \delta_1(H)$. The complete k -graph on r vertices, denoted $K_r^{(k)}$ (or sometimes just K_r) is the k -graph with vertex set $[r]$ and all $\binom{r}{k}$ edges.

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If H is a k -graph and $x \in V(H)$, the *link* of x , written $L_H(x)$, is the $(k-1)$ -graph whose vertex set is $V(H) - \{x\}$ and whose edge set is $N_H(x)$. We write $v(H)$ for $|V(H)|$.

Let G and F be k -graphs. We say that G has a *perfect F -packing* if the vertex set of G can be partitioned into copies of F . Minimum degree conditions that force perfect F -packings in graphs have a long history and have been well studied [1, 11, 21, 23]. In the past decade there has been substantial interest in extending these result to k -graphs [9, 12, 15, 16, 17, 22, 24, 25, 30, 31, 32, 33, 34, 39, 40]. Despite this activity many basic questions in this area remain open. For example, for $k \geq 5$ the minimum degree threshold which forces a perfect matching in k -graphs is not known.

A key ingredient in the proofs of most of the previously cited results are specially designed random-like or quasirandom properties of k -graphs that imply the existence of perfect F -packings. There is a rather well-defined notion of quasirandomness for graphs that originated in early work of Thomason [36, 37] and Chung-Graham-Wilson [7]. These graph quasirandom properties, when generalized to k -graphs, provide a rich structure of inequivalent hypergraph quasirandom properties (see [29, 38]). In [28], the authors studied in detail the packing problem for the simplest of these quasirandom properties, the so-called weak hypergraph quasirandomness. A hypergraph is *linear* if every two edges share at most one vertex. Results of [28] showed that weak hypergraph quasirandomness and an obvious minimum degree condition suffices to obtain perfect F -packings for all linear F , but the result does not hold for certain F that are very close to being linear.

In this paper, we address the packing problem for the other quasirandom properties. A special case of our result identifies what hypergraph quasirandom property and what condition on the link of each vertex is required in order to be able to guarantee a perfect $K_r^{(k)}$ -packing for all r (which implies a perfect F -packing for all F). The quasirandom property naturally has great resemblance to those used in the various (strong) hypergraph regularity lemmas. Keevash's hypergraph blowup lemma [14] has as a corollary that the super-regularity of complexes implies the existence of perfect packings, but our main result below (Theorem 1) shows that a weaker notion of quasirandomness is enough to obtain perfect packings of complete hypergraphs. In fact, we are able to do more: for many of the hypergraph quasirandom properties that have been studied previously in the literature, we give a class of hypergraphs F for which we can find a perfect packing. Before stating Theorem 1, we need to define these notions of hypergraph quasirandomness.

1.1 Notions of Hypergraph Quasirandomness

Our definitions are closely related to the definitions by Towsner [38], which gives the most general treatment of hypergraph quasirandomness.

Definition. Let X be a finite set and let $2^X = \{A : A \subseteq X\}$. An *antichain* is an $\mathcal{I} \subseteq 2^X$ such that $A \subsetneq B$ for all $A, B \in \mathcal{I}$. A *full antichain* is an antichain $\mathcal{I} \subseteq 2^X$ such that $|\mathcal{I}| \geq 2$ and for all $x \in X$, there exists $I \in \mathcal{I}$ with $x \in I$.

Definition. Let $k \geq 1$, let $\mathcal{I} \subseteq 2^{[k]}$ be an antichain, and let H be a k -graph. An \mathcal{I} -layout in H is a tuple of uniform hypergraphs $\Lambda = (\lambda_I)_{I \in \mathcal{I}}$ where λ_I is an $|I|$ -uniform hypergraph

on vertex set $V(H)$. If Λ is an \mathcal{I} -layout, then the k -cliques of Λ , denoted $K_k(\Lambda)$, is the set of all vertex tuples (x_1, \dots, x_k) such that x_1, \dots, x_k are distinct vertices and for each $I \in \mathcal{I}$, $\{x_i : i \in I\} \in \lambda_I$. In an abuse of notation, we will denote by $H \cap K_k(\Lambda)$ the k -tuples (x_1, \dots, x_k) such that $(x_1, \dots, x_k) \in K_k(\Lambda)$ and $\{x_1, \dots, x_k\} \in H$.

We now are ready to define hypergraph quasirandomness.

Definition. Let $0 < \mu, p < 1$. A k -graph H satisfies $\text{Disc}^{(k)}(\mathcal{I}, \geq p, \mu)$ if for every \mathcal{I} -layout Λ ,

$$|H \cap K_k(\Lambda)| \geq p|K_k(\Lambda)| - \mu n^k.$$

The stronger property $\text{Disc}^{(k)}(\mathcal{I}, p, \mu)$ stipulates that for every \mathcal{I} -layout Λ ,

$$\left| |H \cap K_k(\Lambda)| - p|K_k(\Lambda)| \right| \leq \mu n^k.$$

Example. Let $k = 3$ and $\mathcal{I} = \{\{1, 2\}, \{2, 3\}\}$. A 3-graph H satisfies $\text{Disc}^{(3)}(\mathcal{I}, \geq p, \mu)$ if for every two graphs λ_{12} and λ_{23} with vertex set $V(H)$, the number of tuples (x, y, z) with $\{x, y, z\} \in H$, $xy \in \lambda_{12}$, and $yz \in \lambda_{23}$ is at least $p|K_3(\lambda_{12}, \lambda_{23})| - \mu n^3$, where $K_3(\lambda_{12}, \lambda_{23})$ is the set of tuples (x, y, z) with $xy \in \lambda_{12}$ and $yz \in \lambda_{23}$.

Several special cases of this definition deserve mention, since essentially all previously studied hypergraph quasirandomness properties are related to $\text{Disc}^{(k)}(\mathcal{I}, \geq p, \mu)$ for some \mathcal{I} .

- When $\mathcal{I} = \{\{1\}, \dots, \{k\}\}$, then $\text{Disc}^{(k)}(\mathcal{I}, \geq p, \mu)$ is exactly the property $(p, \frac{\mu}{k!})$ -dense from [28] and is closely related to weak quasirandomness studied in [8, 10, 18, 35].
- More generally, when \mathcal{I} is a partition the property $\text{Disc}^{(k)}(\mathcal{I}, p, \mu)$ is essentially the property $\text{Expand}[\pi]$ studied in [26, 27, 29]. In particular, when $\mathcal{I} = \{\{1, \dots, k-1\}, \{k\}\}$, then $\text{Disc}^{(k)}(\mathcal{I}, \geq p, \mu)$ is essentially equivalent to the property considered recently by Keevash (the property called “typical” in [13]) in his recent proof of the existence of designs.
- When $\mathcal{I} = \binom{[k]}{\ell}$, then $\text{Disc}^{(k)}(\mathcal{I}, p, \mu)$ is closely related to the property $\text{CliqueDisc}[\ell]$ studied in [2, 3, 4, 5, 6, 19, 29].
- When $\mathcal{I} = \{I \in \binom{[k]}{k-1} : \{1, \dots, \ell\} \subseteq I\}$, then $\text{Disc}^{(k)}(\mathcal{I}, p, \mu)$ is essentially the same as the property $\text{Deviation}[\ell]$ studied in [4, 5, 3, 19, 29].
- Finally, note that $\text{Disc}^{(k)}(\{\emptyset\}, \geq p, \mu)$ is equivalent to $|H| \geq p \binom{v(H)}{k} - \frac{\mu}{k!} n^k$, since $K_k(\{\emptyset\})$ is the set of all ordered k -tuples of distinct vertices.

Definition. Let $\mathcal{I} \subseteq 2^{[k]}$ be an antichain. A k -graph F is \mathcal{I} -adapted if there exists an ordering E_1, \dots, E_m of the edges of F and bijections $\phi_i : E_i \rightarrow [k]$ such that for each $1 \leq j < i \leq m$, the following holds: there exists an $I \in \mathcal{I}$ with $\{\phi_i(x) : x \in E_j \cap E_i\} \subseteq I \in \mathcal{I}$.

In words, F is \mathcal{I} -adapted if the set of labels assigned to E_i which appear on $E_j \cap E_i$ is a subset of a set in \mathcal{I} .

Let $\mathcal{I} \subseteq 2^{[k]}$ and $\mathcal{J} \subseteq 2^{[k-1]}$ be antichains. A k -graph F is $(\mathcal{I}, \mathcal{J})$ -adapted if F is \mathcal{I} -adapted and there exists $x \in V(F)$, an ordering E_1, \dots, E_m of the edges of F , and bijections $\psi_i : E_i \rightarrow [k]$ such that for all $1 \leq j < i \leq m$, the following holds.

- If $x \notin E_i$ then there exists $I \in \mathcal{I}$ with $\{\psi_i(y) : y \in E_j \cap E_i\} \subseteq I$.
- If $x \in E_i$ then $\psi_i(x) = k$ and there exists $J \in \mathcal{J}$ with $\{\psi_i(y) : y \in E_j \cap E_i, y \neq x\} \subseteq J$.

1.2 Our Results

The following is our main result.

Theorem 1. Let $k \geq 2$, $\mathcal{I} \subseteq 2^{[k]}$ be a full antichain, $\mathcal{J} \subseteq 2^{[k-1]}$, and $0 < \alpha, p < 1$. For every $(\mathcal{I}, \mathcal{J})$ -adapted k -graph F , there exists $\mu > 0$ and n_0 so that the following holds. Let H be an n -vertex k -graph where $n \geq n_0$ and $v(F) | n$. Suppose that H satisfies $\text{Disc}^{(k)}(\mathcal{I}, \geq p, \mu)$ and that $L_H(x)$ satisfies $\text{Disc}^{(k-1)}(\mathcal{J}, \geq \alpha, \mu)$ for all $x \in V(H)$. Then H has a perfect F -packing.

It is straightforward to see that if \mathcal{I} and \mathcal{I}' are such that for every $I' \in \mathcal{I}'$, there exists $I \in \mathcal{I}$ with $I' \subseteq I$, then $\text{Disc}^{(k)}(\mathcal{I}, \geq p, \mu) \Rightarrow \text{Disc}^{(k)}(\mathcal{I}', \geq p, \mu)$. Also, if $\mathcal{I} = \binom{[k]}{k-1}$ and $\mathcal{J} = \binom{[k-1]}{k-2}$, then every F is $(\mathcal{I}, \mathcal{J})$ -adapted. Thus to find the weakest quasirandom condition to apply Theorem 1 to a given k -graph F , one should find the minimal \mathcal{I} and \mathcal{J} for which F is $(\mathcal{I}, \mathcal{J})$ -adapted. For example, if $C = \{abc, bcd, def, aef\}$, then C is $(\mathcal{I}, \mathcal{J})$ -adapted where $\mathcal{I} = \{\{1, 2\}, \{3\}\}$ and $\mathcal{J} = \{\emptyset\}$ (let $x = a$ and order the edges which contain a first).

As mentioned above, special cases of $\text{Disc}^{(k)}(\mathcal{I}, \geq p, \mu)$ correspond to previously studied quasirandom properties so that Theorem 1 generalizes several previous results.

- Let $k = 2$. The only full antichain is $\mathcal{I} = \{\{1\}, \{2\}\}$. For this \mathcal{I} , all graphs F are $(\mathcal{I}, \mathcal{J})$ -adapted if $\mathcal{J} = \{\emptyset\}$. To see this, pick $x \in V(F)$ and place all edges incident to x first in the ordering for the definition of $(\mathcal{I}, \mathcal{J})$ -adapted. Now the property $\text{Disc}^{(2)}(\mathcal{I}, \geq p, \mu)$ just states that G is quasirandom (in fact only “one-sided” quasirandom). Also, the condition “ $L_H(x)$ satisfies $\text{Disc}^{(1)}(\{\emptyset\}, \geq \alpha, \mu)$ for every $x \in V(H)$ ” is equivalent to the condition that $\delta(H) \geq (\alpha - \mu)(n - 1)$. To see this, recall from before that if H' is an r -graph the property “ H' satisfies $\text{Disc}^{(1)}(\{\emptyset\}, \geq \alpha, \mu)$ ” is equivalent to the property that $|H'| \geq \alpha \binom{v(H')}{r} - \frac{\mu}{r!} v(H')^r$. Thus Theorem 1 for $k = 2$ states that if G is an n -vertex quasirandom graph, $v(F) | n$, and $\delta(G) \geq (\alpha - \mu)(n - 1)$, then G has a perfect F -packing. This fact is a simple consequence of the blowup lemma of Komlós-Sárközy-Szemerédi [20].
- For $k \geq 2$ with \mathcal{I} a partition into singletons, we obtain exactly [28, Theorem 3]. In this case, $\text{Disc}^{(k)}(\mathcal{I}, \geq p, \mu)$ is equivalent to $(p, \frac{\mu}{k!})$ -dense from [28], an \mathcal{I} -adapted k -graph is a linear k -graph, and one can take $\mathcal{J} = \{\emptyset\}$. Similar to the previous paragraph, the condition “ $L_H(x)$ satisfies $\text{Disc}^{(k-1)}(\{\emptyset\}, \geq \alpha, \mu)$ for every $x \in V(H)$ ” is equivalent to the condition that $\delta(H) \geq \alpha \binom{v(H)-1}{k-1} - \frac{\mu}{(k-1)!} v(H)^{k-1}$.

- If $\mathcal{I} = \binom{[k]}{k-1}$ and $\mathcal{J} = \binom{[k-1]}{k-2}$ then every k -graph F is $(\mathcal{I}, \mathcal{J})$ -adapted. Thus Theorem 1 implies the following corollary.

Corollary 2. *Fix $2 \leq k \leq r$. For every $0 < \alpha, p < 1$, there exists $\mu > 0$ and n_0 such that the following holds. Let H be an n -vertex k -graph with $n \geq n_0$ and $r|n$. If H satisfies $\text{Disc}^{(k)}(\binom{[k]}{k-1}, \geq p, \mu)$ and $L_H(x)$ satisfies $\text{Disc}^{(k-1)}(\binom{[k-1]}{k-2}, \geq \alpha, \mu)$ for every $x \in V(H)$, then H has a perfect $K_r^{(k)}$ -packing.*

Keavash's hypergraph blowup lemma [14] also guarantees perfect $K_r^{(k)}$ -packings under certain regularity conditions, however the hypotheses of Corollary 2 are slightly weaker. Indeed, the main extra requirement that [14] places on H is [14, Definition 3.16 part (iii)]; translated into our language, for 3-graphs this property says roughly that for every $x \in V(H)$, if W is a set of triples where each triple contains some pair from $L_H(x)$, then $|H \cap W| \approx p|W|$.

Next, we investigate if either of the conditions $\text{Disc}^{(k)}(\mathcal{I}, \geq p, \mu)$ or $\text{Disc}^{(k-1)}(\mathcal{J}, \geq \alpha, \mu)$ in the links from Theorem 1 can be weakened. This question was studied by the authors [28] in detail when \mathcal{I} is a partition, and it turns out that for certain non-linear F it is possible to weaken the conditions (see [28] for details). Most likely, the constructions and results from [28] can be generalized to all \mathcal{I} . In this paper, we focus only on the case $\mathcal{I} = \binom{[k]}{k-1}$ and $\mathcal{J} = \binom{[k-1]}{k-2}$, which corresponds to the condition required for perfect $K_r^{(k)}$ -packings. In this case, neither condition can be weakened, so that Theorem 1 cannot be improved in general.

Proposition 3. *For every $k \geq 3$ there exists an r (depending only on k) such that the following holds. Let $\alpha = p = \frac{k-1}{k}$ and let $\mathcal{I} \subseteq 2^{[k]}$ be a full antichain where $\mathcal{I} \neq \binom{[k]}{k-1}$. For every $\mu > 0$, there exists n_0 such that for all $n \geq n_0$ there exists an n -vertex k -graph H which*

- *satisfies $\text{Disc}^{(k)}(\mathcal{I}, \geq p, \mu)$,*
- *fails $\text{Disc}^{(k)}(\binom{[k]}{k-1}, \geq p, \mu)$,*
- *for every $x \in V(H)$ the link $L_H(x)$ satisfies $\text{Disc}^{(k-1)}(\binom{[k-1]}{k-2}, \geq \alpha, \mu)$,*
- *has no copy of K_r (so no perfect K_r -packing).*

Proposition 4. *For every $k \geq 3$ there exists an r (depending only on k) such that the following holds. Let $\alpha = p = \frac{k-1}{k}$ and let $\mathcal{J} \subseteq 2^{[k-1]}$ be a full antichain where $\mathcal{J} \neq \binom{[k-1]}{k-2}$. For every $0 < \mu, p < 1$, there exists n_0 such that for all $n \geq n_0$ with $r|n$, there exists an n -vertex k -graph H which*

- *satisfies $\text{Disc}^{(k)}(\binom{[k]}{k-1}, \geq p, \mu)$,*
- *for every $x \in V(H)$ the link $L_H(x)$ satisfies $\text{Disc}^{(k-1)}(\mathcal{J}, \geq \alpha, \mu)$,*
- *there exists $x \in V(H)$ such that the link $L_H(x)$ fails $\text{Disc}^{(k-1)}(\binom{[k-1]}{k-2}, \geq \alpha, \mu)$,*

- has no perfect K_r -packing.

The remainder of this paper is organized as follows. In Sections 2 and 3 we discuss the two main tools needed for the proof of Theorem 1, in Section 4 we prove Theorem 1, and finally in Section 5 we explain the constructions which prove Propositions 3 and 4.

2 Absorbing Sets

One of the main tools for our proof of Theorem 1 is the absorbing technique of Rödl-Ruciński-Szemerédi [34]. We will use the following absorbing lemma from [28] without modification.

Definition. Let F and H be k -graphs and let $A, B \subseteq V(H)$. We say that A F -absorbs B or that A is an F -absorbing set for B if both $H[A]$ and $H[A \cup B]$ have perfect F -packings. When F is a single edge, we say that A edge-absorbs B .

Definition. Let F and H be k -graphs, $\epsilon > 0$, and a and b be multiples of $v(F)$. We say that H is (a, b, ϵ, F) -rich if for all $B \in \binom{V(H)}{b}$ there are at least ϵn^a sets in $\binom{V(H)}{a}$ which F -absorb B .

Lemma 5. (Absorbing Lemma, specialized version of [28, Lemma 10]) Let F be a k -graph, $\epsilon > 0$, and a and b be multiples of $v(F)$. There exists an n_0 and $\omega > 0$ such that for all n -vertex k -graphs H with $n \geq n_0$, the following holds. If H is (a, b, ϵ, F) -rich, then there exists an $A \subseteq V(H)$ such that $a \parallel |A|$ and A F -absorbs all sets C satisfying the following conditions: $C \subseteq V(H) - A$, $|C| \leq \omega n$, and $b \parallel |C|$.

3 Embedding Lemma

Definition. Let $k \geq 2$ and $0 \leq m \leq f$. Let F and H be k -graphs with $V(F) = \{w_1, \dots, w_f\}$. A labeled copy of F in H is an edge-preserving injection from $V(F)$ to $V(H)$. A degenerate labeled copy of F in H is an edge-preserving map from $V(F)$ to $V(H)$ that is not an injection. Let $1 \leq m \leq f$ and let $Z_1, \dots, Z_m \subseteq V(H)$. Set $\text{inj}[F \rightarrow H; w_1 \rightarrow Z_1, \dots, w_m \rightarrow Z_m]$ to be the number of edge-preserving injections $\psi : V(F) \rightarrow V(H)$ such that $\psi(w_i) \in Z_i$ for all $1 \leq i \leq m$. If $Z_i = \{z_i\}$, we abbreviate $w_i \rightarrow \{z_i\}$ as $w_i \rightarrow z_i$.

The embedding lemma (Lemma 6) proved in this section shows that if H satisfies $\text{Disc}^{(k)}(\mathcal{I}, \geq p, \mu)$ and $\text{Disc}^{(k-1)}(\mathcal{J}, \geq \alpha, \mu)$ in the links, then H contains many copies of F if F is $(\mathcal{I}, \mathcal{J})$ -adapted. In fact, it says more: if m of the vertices of F are pre-specified and F satisfies the following more technical condition, then there are many copies of F using the m pre-specified vertices.

Definition. Let $k \geq 2$, $\mathcal{I} \subseteq 2^{[k]}$ and $\mathcal{J} \subseteq 2^{[k-1]}$ be antichains, F a k -graph, and $s_1, \dots, s_m \in V(F)$. We say that F is $(\mathcal{I}, \mathcal{J})$ -adapted at s_1, \dots, s_m if there exists an ordering E_1, \dots, E_t of the edges of F such that

- for every i , $|E_i \cap \{s_1, \dots, s_m\}| \leq 1$,
- for every E_i with $E_i \cap \{s_1, \dots, s_m\} = \emptyset$, there exists a bijection $\phi_i : E_i \rightarrow [k]$ such that for all $j < i$, there exists $I \in \mathcal{I}$ with $\{\phi_i(x) : x \in E_j \cap E_i\} \subseteq I$,
- for every E_i with $s_\ell \in E_i$, there exists a bijection $\psi_i : E_i \setminus \{s_\ell\} \rightarrow [k-1]$ such that for all $j < i$, there exists $J \in \mathcal{J}$ with $\{\psi_i(x) : x \in E_j \cap E_i, x \neq s_\ell\} \subseteq J$.

Note that $m = 0$ is possible, in which case the definition is equivalent to \mathcal{I} -adapted.

Lemma 6. Let $k \geq 2$, $0 < \alpha, \gamma, p < 1$, and $\mathcal{I} \subseteq 2^{[k]}$ and $\mathcal{J} \subseteq 2^{[k-1]}$ be antichains. Let F be an f -vertex k -graph with $V(F) = \{s_1, \dots, s_m, t_{m+1}, \dots, t_f\}$. Suppose that F is $(\mathcal{I}, \mathcal{J})$ -adapted at s_1, \dots, s_m . Then there exists an n_0 and $\mu > 0$ such that the following is true.

Let H be an n -vertex k -graph with $n \geq n_0$, where H satisfies $\text{Disc}^{(k)}(\mathcal{I}, \geq p, \mu)$. If $m > 0$, then also assume that $L_H(x)$ satisfies $\text{Disc}^{(k-1)}(\mathcal{J}, \geq \alpha, \mu)$ for every vertex $x \in V(H)$. Let $y_1, \dots, y_m \in V(H)$ be distinct and let $V_{m+1}, \dots, V_f \subseteq V(H)$. Then

$$\begin{aligned} \text{inj}[F \rightarrow H; s_1 \rightarrow y_1, \dots, s_m \rightarrow y_m, t_{m+1} \rightarrow V_{m+1}, \dots, t_f \rightarrow V_f] \\ \geq \alpha^{d_F(s_1)} \dots \alpha^{d_F(s_m)} p^{|F| - \sum d_F(s_i)} |V_{m+1}| \dots |V_f| - \gamma n^{f-m}. \end{aligned}$$

Proof. We first prove the lemma under the additional assumption that the sets V_{m+1}, \dots, V_f are pairwise disjoint. This is proved by induction on $|F|$. If $|F| = 0$, then

$$\begin{aligned} \text{inj}[F \rightarrow H; s_1 \rightarrow y_1, \dots, s_m \rightarrow y_m, t_{m+1} \rightarrow V_{m+1}, \dots, t_f \rightarrow V_f] &\geq \prod_{i=m+1}^f (|V_i| - f) \\ &\geq \alpha^0 p^0 \prod_{i=m+1}^f |V_i| - \gamma n^{f-m} \end{aligned}$$

for large n . So assume F has at least one edge and let E be the last edge in an ordering of the edges of F which witness that F is $(\mathcal{I}, \mathcal{J})$ -adapted at s_1, \dots, s_m . (Recall that if $m = 0$ then $(\mathcal{I}, \mathcal{J})$ -adapted at s_1, \dots, s_m is equivalent to \mathcal{I} -adapted.)

Let F_* be the hypergraph formed by deleting all vertices of E from F . Let F_- be the hypergraph formed by removing the edge E from F but keeping the same vertex set. Let Q_* be an injective edge-preserving map $Q_* : V(F_*) \rightarrow V(H)$ where $Q_*(s_i) = y_i$ for $1 \leq i \leq m$ and $Q_*(t_j) \in V_j$ for $t_j \notin E$. There are two cases.

Case 1: $E \cap \{s_1, \dots, s_m\} = \emptyset$. Let $\phi : E \rightarrow [k]$ be the bijection from the definition of $(\mathcal{I}, \mathcal{J})$ -adapted at s_1, \dots, s_m and assume the vertices of F are labeled such that $E = \{t_{m+1}, \dots, t_{m+k}\}$, where $\phi(t_{m+i}) = i$. For each $I \in \mathcal{I}$, define an $|I|$ -uniform hypergraph λ_{I, Q_*} with vertex set $V(H)$ as follows. Let $I = \{i_1, \dots, i_{|I|}\}$. Make $\{z_{i_1}, \dots, z_{i_{|I|}}\} \in \binom{V(H)}{|I|}$ a hyperedge of λ_{I, Q_*} if $z_{i_j} \in V_{m+i_j}$ for all j and when the map Q_* is extended to map t_{i_j} to z_{i_j} for all j , this extended map is an edge-preserving map from $F_-[V(F_*) \cup \{t_{i_1}, \dots, t_{i_{|I|}}\}]$ to H . More informally, λ_{I, Q_*} consists of all $|I|$ -sets to which Q_* can be extended to produce a copy of F_* together with the vertices of E indexed by I . Let $\Lambda_{Q_*} = (\lambda_{I, Q_*})_{I \in \mathcal{I}}$.

Now, if $(z_{m+1}, \dots, z_{m+k})$ is a k -tuple in $K_k(\Lambda_{Q_*})$, then the map Q_* can be extended to map t_j to z_j for $m+1 \leq j \leq m+k$ to produce an edge-preserving map from F_- to H . To see this, let E' be an edge of F_- . Since E is the last edge in the ordering, if $E' \cap E = \{t_{j_1}, \dots, t_{j_r}\}$ then there exists some $I \in \mathcal{I}$ with $\{j_1, \dots, j_r\} \subseteq I$ since F is \mathcal{I} -adapted. Since $(z_{m+1}, \dots, z_{m+k})$ is a k -clique, $\{z_{m+i} : i \in I\} \in \lambda_{I, Q_*}$. This implies that there is some permutation η of I such that extending Q_* to map t_{m+i} to $z_{m+\eta(i)}$ produces an edge-preserving map. Since the V_{m+i} s are pairwise disjoint and $z_{m+i} \in V_{m+i}$ for all $i \in I$, η must be the identity permutation, i.e. extending the map Q_* to map t_{m+i} to z_{m+i} for all $i \in I$ produces an edge-preserving map. Thus extending the map Q_* to map t_{j_p} to z_{j_p} for all p is an edge-preserving map and E' is one of the preserved edges. Finally, since the V_j s are disjoint, each k -tuple in $K_k(\Lambda_{Q_*})$ corresponds to exactly one labeled copy of F_- in H which extend Q_* with t_j mapped into V_j for all j . Similarly, $|H \cap K_k(\Lambda_{Q_*})|$ is exactly the number of labeled copies of F in H which extend Q_* with t_j mapped into V_j for all j . Thus,

$$\begin{aligned} \text{inj}[F \rightarrow H; s_1 \rightarrow y_1, \dots, s_m \rightarrow y_m, t_{m+1} \rightarrow V_{m+1}, \dots, t_f \rightarrow V_f] &= \sum_{Q_*} |H \cap K_k(\Lambda_{Q_*})| \\ \text{inj}[F_- \rightarrow H; s_1 \rightarrow y_1, \dots, s_m \rightarrow y_m, t_{m+1} \rightarrow V_{m+1}, \dots, t_f \rightarrow V_f] &= \sum_{Q_*} |K_k(\Lambda_{Q_*})|. \end{aligned} \quad (1)$$

Since H satisfies $\text{Disc}^{(k)}(\mathcal{I}, \geq p, \mu)$,

$$\begin{aligned} \text{inj}[F \rightarrow H; s_1 \rightarrow y_1, \dots, s_m \rightarrow y_m, t_{m+1} \rightarrow V_{m+1}, \dots, t_f \rightarrow V_f] \\ \geq \sum_{Q_*} (p|K_k(\Lambda_{Q_*})| - \mu n^k) \\ \geq p \sum_{Q_*} |K_k(\Lambda_{Q_*})| - \mu n^{f-m}, \end{aligned} \quad (2)$$

where the last inequality is because there are at most n^{f-m-k} maps Q_* , since F_* has $f-k$ vertices and $s_i \in V(F_*)$ must map to y_i . Combining (1) and (2) and then applying induction,

$$\begin{aligned} \text{inj}[F \rightarrow H; s_1 \rightarrow y_1, \dots, s_m \rightarrow y_m, t_{m+1} \rightarrow V_{m+1}, \dots, t_f \rightarrow V_f] \\ \geq p \text{inj}[F_- \rightarrow H; s_1 \rightarrow y_1, \dots, s_m \rightarrow y_m, t_{m+1} \rightarrow V_{m+1}, \dots, t_f \rightarrow V_f] - \mu n^{f-m} \\ \geq p (\alpha^{\sum d(s_i)} p^{|F|-1-\sum d(s_i)} |V_{m+1}| \cdots |V_f| - \gamma n^{f-m}) - \mu n^{f-m}. \end{aligned}$$

Let $\mu = (1-p)\gamma$ so that the proof of this case complete.

Case 2: $s_\ell \in E$. (Since F is $(\mathcal{I}, \mathcal{J})$ -adapted at s_1, \dots, s_m , at most one vertex s_ℓ can be in E .) Let $\psi : E \setminus \{s_\ell\} \rightarrow [k-1]$ be the bijection from the definition of $(\mathcal{I}, \mathcal{J})$ -adapted at s_1, \dots, s_m and assume the vertices of E are labeled such that $E = \{s_\ell, t_{m+1}, \dots, t_{m+k-1}\}$ where $\psi(t_{m+j}) = j$. This case is very similar to the previous case, except we will use $\text{Disc}^{(k-1)}(\mathcal{J}, \geq \alpha, \mu)$ in the link of y_ℓ . For each $J \in \mathcal{J}$, define a $|J|$ -uniform hypergraph λ_{J, Q_*} with vertex set $V(H)$ as follows. Let $J = \{j_1, \dots, j_{|J|}\}$. Make $\{z_{j_1}, \dots, z_{j_{|J|}}\}$ a hyperedge of λ_{J, Q_*} if $z_{j_r} \in V_{j_r}$ for all r and extending the map Q_* to map s_ℓ to y_ℓ and mapping t_{j_r}

to z_{j_r} for all r produces an edge-preserving map. Let $\Lambda_{Q_*} = (\lambda_{J, Q_*})_{J \in \mathcal{J}}$. Similar to before, if $(z_{m+1}, \dots, z_{m+k-1})$ is a $(k-1)$ -tuple in $K_{k-1}(\Lambda_{Q_*})$, then the map Q_* can be extended to map s_ℓ to y_ℓ and map t_i to z_i for $m+1 \leq i \leq m+k-1$ to produce an edge-preserving map from F_- to H . Thus $|K_{k-1}(\Lambda_{Q_*})|$ is exactly the number of labeled copies of F_- in H which extend Q_* . Similarly, $|L_H(y_\ell) \cap K_{k-1}(\Lambda_{Q_*})|$ is exactly the number of labeled copies of F in H which extend Q_* .

Now formulas similar to (1) and (2) and the fact that $L_H(y_\ell)$ satisfies $\text{Disc}^{(k-1)}(\mathcal{J}, \geq \alpha, \mu)$ completes this case. This concludes the proof of the lemma if the sets V_{m+1}, \dots, V_f are pairwise disjoint.

Now assume that the sets V_{m+1}, \dots, V_f are not necessarily pairwise disjoint. Let $\mathcal{P} = \{(P_{m+1}, \dots, P_f) : P_{m+1}, \dots, P_f \text{ is a partition of } V(H)\}$ so that $|\mathcal{P}| = (f-m)^n$. Now

$$\begin{aligned} & \text{inj}[F \rightarrow H; s_1 \rightarrow y_1, \dots, s_m \rightarrow y_m, t_{m+1} \rightarrow V_{m+1}, \dots, t_f \rightarrow V_f] \\ &= \frac{1}{(f-m)^{n-f+m}} \sum_{(P_{m+1}, \dots, P_f) \in \mathcal{P}} \text{inj}[F \rightarrow H; s_1 \rightarrow y_1, \dots, s_m \rightarrow y_m, \\ & \quad t_{m+1} \rightarrow V_{m+1} \cap P_{m+1}, \dots, t_f \rightarrow V_f \cap P_f]. \end{aligned}$$

Indeed, each labeled copy of F of the right form will be counted exactly $(f-m)^{n-f+m}$ times by the sum over all partitions, since the images of t_{m+1}, \dots, t_f must map into the cooresponding part of the partition and all other vertices of H can be distributed to any of the parts of the partition. Let $\delta = \alpha^{d_F(s_1)} \dots \alpha^{d_F(s_m)} p^{|F| - \sum d_F(s_i)}$. Since $V_{m+1} \cap P_{m+1}, \dots, V_f \cap P_f$ are pairwise disjoint,

$$\begin{aligned} & \text{inj}[F \rightarrow H; s_1 \rightarrow y_1, \dots, s_m \rightarrow y_m, t_{m+1} \rightarrow V_{m+1}, \dots, t_f \rightarrow V_f] \\ & \geq \frac{1}{(f-m)^{n-f+m}} \sum_{(P_{m+1}, \dots, P_f) \in \mathcal{P}} (\delta |V_{m+1} \cap P_{m+1}| \dots |V_f \cap P_f| - \gamma n^{f-m}) \\ & = \delta |V_{m+1}| \dots |V_f| - \frac{\gamma n^{f-m} |\mathcal{P}|}{(f-m)^{n-f+m}} \geq \delta |V_{m+1}| \dots |V_f| - \gamma n^{f-m}. \end{aligned}$$

□

4 Packing $(\mathcal{I}, \mathcal{J})$ -adapted hypergraphs

In this section we prove Theorem 1. The proof has several stages: we first prove that the quasirandom conditions on H imply that H is rich, then we use Lemma 5 to set aside a vertex set A which can absorb all reasonably sized sets, next we use the embedding lemma (Lemma 6) to produce an almost perfect packing in $H - A$, and finally we use the properties of A to absorb the remaining vertices.

4.1 Richness

In this subsection, we prove that the conditions on H in Theorem 1 imply that H is $(f^2 - f, f, \epsilon, F)$ -rich, where $f = v(F)$.

Lemma 7. *Let $k \geq 2$, $\mathcal{I} \subseteq 2^{[k]}$ be a full antichain, and $\mathcal{J} \subseteq 2^{[k-1]}$ an antichain. Let F be an $(\mathcal{I}, \mathcal{J})$ -adapted k -graph with f vertices. For every $0 < \alpha, p < 1$, there exists $\mu, \epsilon > 0$ and n_0 so that the following holds. Let H be an n -vertex k -graph where $n \geq n_0$. Also, assume that H satisfies $\text{Disc}^{(k)}(\mathcal{I}, \geq p, \mu)$ and that $L_H(z)$ satisfies $\text{Disc}^{(k-1)}(\mathcal{J}, \geq \alpha, \mu)$ for every vertex $z \in V(H)$. Then H is $(f^2 - f, f, \epsilon, F)$ -rich.*

Proof. Let $a = f(f - 1)$ and $b = f$. Our task is to come up with an $\epsilon > 0$ such that for large n and all $B \in \binom{V(H)}{b}$, there are at least ϵn^a vertex sets of size a which F -absorb B ; we will define ϵ and μ later. Let $V(F) = \{w_0, \dots, w_{f-1}\}$, where w_0 is the special vertex in the definition that F is $(\mathcal{I}, \mathcal{J})$ -adapted.

Next, form the following k -graph F' . Let

$$V(F') = \{x_{i,j} : 0 \leq i, j \leq f - 1\}.$$

(We think of the vertices of F' as arranged in a grid with i as the row and j as the column.) Form the edges of F' as follows: for each fixed $1 \leq i \leq f - 1$, let $\{x_{i,0}, \dots, x_{i,f-1}\}$ induce a copy of F where $x_{i,j}$ is mapped to w_j . Similarly, for each fixed $0 \leq j \leq f - 1$, let $\{x_{0,j}, \dots, x_{f-1,j}\}$ induce a copy of F where $x_{i,j}$ is mapped to w_i . Note that we therefore have a copy of F in each column and a copy of F in each row besides the zeroth row.

Now fix $B = \{b_0, \dots, b_{f-1}\} \subseteq V(H)$; we want to show that B is F -absorbed by many a -sets. Note that any labeled copy of F' in H which maps $x_{0,0} \rightarrow b_0, \dots, x_{0,f-1} \rightarrow b_{f-1}$ produces an F -absorbing set for B as follows. Let $Q : V(F') \rightarrow V(H)$ be an edge-preserving injection where $Q(b_j) = x_{0,j}$ (so Q is a labeled copy of F' in H where the set B is the zeroth row of F'). Let $A = \{Q(x_{i,j}) : 1 \leq i \leq f - 1, 0 \leq j \leq f - 1\}$ consist of all vertices in rows 1 through $f - 1$. Then A has a perfect F -packing consisting of the copies of F on the rows, and $A \cup B$ has a perfect F -packing consisting of the copies of F on the columns. Therefore, A F -absorbs B .

To complete the proof, we therefore just need to use Lemma 6 where $m = f$ and $s_1 = x_{0,0}, \dots, s_f = x_{0,f-1}$ to show that there are many copies of F' with B as the zeroth row. To do so, we need to show that F' is $(\mathcal{I}, \mathcal{J})$ -adapted at s_1, \dots, s_m . Indeed, consider the following ordering of edges of F' . First, list the edges of F' in the first column, then the edges of F' in the second column, and so on until the k th column. Next, list the edges of F' in the first row, then the second row, and so on until the $(k - 1)$ st row. Within each row or column, list the edges in the ordering given in the definition of F being $(\mathcal{I}, \mathcal{J})$ -adapted. For the bijections ϕ or ψ , use the same bijection as in the definition of F being $(\mathcal{I}, \mathcal{J})$ -adapted. Now consider $E_i, E_j \in F'$ in this ordering with $j < i$. If E_i and E_j are from the same row or the same column, then since F is $(\mathcal{I}, \mathcal{J})$ -adapted the condition on $E_i \cap E_j$ is satisfied. If E_i and E_j are in different rows or columns, the size of their intersection is at most one. If $E_i \cap E_j = \emptyset$ then the condition is trivially satisfied. If $E_i \cap E_j = \{u\}$, then E_i must be from a row since $i > j$. Then E_i does not contain any s_1, \dots, s_m , so we must show that there is some $I \in \mathcal{I}$ so that $\phi_i(u) \in I$. This is true because \mathcal{I} is full. Thus F' is $(\mathcal{I}, \mathcal{J})$ -adapted at s_1, \dots, s_m .

Now apply Lemma 6 to F' with $m = f$, $s_1 = x_{0,0}, \dots, s_f = x_{0,f-1}$, $V_{m+1} = \dots = V_{f^2} = V(H) - B$, and $\gamma = \frac{1}{2}\alpha \sum d(x_{0,j}) p^{|F| - \sum d(x_{0,j})}$. Ensure that n_0 is large enough and μ is small

enough apply Lemma 6 to show that

$$\text{inj}[F' \rightarrow H; x_{0,0} \rightarrow b_0, \dots, x_{0,f-1} \rightarrow b_{f-1}] \geq \gamma \left(\frac{n}{2}\right)^{f^2-f} = \frac{\gamma}{2^{f^2-f}} n^a.$$

Each labeled copy of F' produces a labeled F -absorbing set for B , so there are at least $\frac{\gamma}{a!2^{f^2-f}} n^a$ F -absorbing sets for B . The proof is complete by letting $\epsilon = \frac{\gamma}{a!2^{f^2-f}}$. \square

4.2 Almost perfect packings

In this section we prove that the conditions in Theorem 1 imply that there exists a perfect F -packing covering almost all the vertices of H .

Lemma 8. *Let $k \geq 2$ and $\mathcal{I} \subseteq 2^{[k]}$ be a full antichain. Fix $0 < p < 1$ and an \mathcal{I} -adapted k -graph F with f vertices. Fix an integer b with $f|b$. For any $0 < \omega < 1$, there exists n_0 and $\mu > 0$ such that the following holds. Let H be an k -graph satisfying $\text{Disc}^{(k)}(\mathcal{I}, \geq p, \mu)$ with $n \geq n_0$ and $f|n$. Then there exists $C \subseteq V(H)$ such that $|C| \leq \omega n$, $b||C|$, and $H[\bar{C}]$ has a perfect F -packing.*

Proof. First, select n_0 large enough and μ small enough so that any vertex set C of size $\lceil \frac{\omega}{2} \rceil$ contains a copy of F . To see this, let $\gamma = \frac{1}{2} p^{|F|} (\frac{\omega}{2})^f$ and select n_0 and $\mu > 0$ according to Lemma 6 with $m = 0$. (Recall that if $m = 0$ then the condition $(\mathcal{I}, \mathcal{J})$ -adapted on F at \emptyset just reduces to the statement that F is \mathcal{I} -adapted.) Now if $C \subseteq V(H)$ with $|C| \geq \frac{\omega}{2} n$, then let $V_1 = \dots = V_f = C$ so that $|V_i| \geq \frac{\omega}{2}$ for all i . Then Lemma 6 implies there are at least $p^{|F|} \prod |V_i| - \gamma n^f \geq p^{|F|} (\frac{\omega}{2})^f n^f - \gamma n^f = \gamma n^f > 0$ copies of F inside C .

Now let F_1, \dots, F_t be a greedily constructed F -packing. That is, F_1, \dots, F_t are disjoint copies of F and $C := V(H) - V(F_1) - \dots - V(F_t)$ has no copy of F . By the previous paragraph, $|C| \leq \frac{\omega}{2} n$. Since $f|n$ and $H[\bar{C}]$ has a perfect F -packing, $f||C|$. Thus we can let $y \equiv -\frac{|C|}{f} \pmod{b}$ with $0 \leq y < b$ and take y of the copies of F in the F -packing of $H[\bar{C}]$ and add their vertices into C so that $b||C|$. \square

4.3 Proof of Theorem 1

Proof of Theorem 1. First, apply Lemma 7 to produce $\epsilon > 0$. Next, select $\omega > 0$ according to Lemma 5 and $\mu_1 > 0$ according to Lemma 8. Also, make n_0 large enough so that both Lemma 5 and 8 can be applied. Let $\mu = \mu_1 \omega^k$. All the parameters have now been chosen.

By Lemmas 5 and 7, there exists a set $A \subseteq V(H)$ such that A F -absorbs C for all $C \subseteq V(H) \setminus A$ with $|C| \leq \omega n$ and $b ||C|$. If $|A| \geq (1 - \omega)n$, then A F -absorbs $V(H) \setminus A$ so that H has a perfect F -packing. Thus $|A| \leq (1 - \omega)n$. Next, let $H' := H[\bar{A}]$ and notice that H' satisfies $\text{Disc}^{(k)}(\mathcal{I}, \geq p, \mu_1)$ since $v(H') \geq \omega n$ and

$$\mu n^k \leq \frac{\mu}{\omega^k} v(H')^k = \mu_1 v(H')^k.$$

Therefore, by Lemma 8, there exists a vertex set $C \subseteq V(H') = V(H) \setminus A$ such that $|C| \leq \omega n$, $|C|$ is a multiple of b , and $H'[\bar{C}]$ has a perfect F -packing. Now Lemma 5 implies that A

F -absorbs C . The perfect F -packing of $A \cup C$ and the perfect F -packing of $H'[\bar{C}]$ produces a perfect F -packing of H . \square

5 Constructions

In this section, we prove Propositions 3 and 4 using the following construction.

Construction. Let $k \geq 2$. Let $A_n^{(k)}$ be the following probability distribution over n -vertex k -graphs. Let $f : \binom{V(A_n^{(k)})}{k-1} \rightarrow \{0, \dots, k-1\}$ be a random k -coloring of the $(k-1)$ -sets. Make $E \in \binom{V(A_n^{(k)})}{k}$ an edge of $A_n^{(k)}$ if

$$\sum_{\substack{T \subseteq E \\ |T|=k-1}} f(T) \not\equiv 0 \pmod{k}.$$

Lemma 9. Let $p = \frac{k-1}{k}$ and $\epsilon > 0$. Then with probability going to one as n goes to infinity,

$$\left| |A_n^{(k)}| - p \binom{n}{k} \right| < \epsilon n^k.$$

Proof. Each k -set is an edge with probability exactly p , so $\mathbb{E}[|A_n^{(k)}|] = p \binom{n}{k}$. A simple second moment argument then shows that with high probability the number of edges is concentrated around $p \binom{n}{k}$. \square

Lemma 10. There exists a μ_0 such that for all $0 < \mu < \mu_0$, with probability going to one as n goes to infinity, $A_n^{(k)}$ fails $\text{Disc}^{(k)}(\binom{[k]}{k-1}, \geq p, \mu)$.

Proof. Let Z be the $(k-1)$ -graph whose edges are all the $(k-1)$ -sets colored zero. Let $\Lambda = (Z, \dots, Z)$ be the $\binom{[k]}{k-1}$ -layout consisting of Z in every coordinate. Now any k -clique (z_1, \dots, z_k) of Λ is not a hyperedge of $A_n^{(k)}$, since every $(k-1)$ -subset of $\{z_1, \dots, z_k\}$ has color zero. This Λ will show that $A_n^{(k)}$ fails $\text{Disc}^{(k)}(\binom{[k]}{k-1}, \geq p, \mu)$ if $|K_k(\Lambda)|$ is large enough. Each k -tuple of vertices is a k -clique with probability $(\frac{1}{k})^k$, so $\mathbb{E}[|K_k(\Lambda)|] = k^{-k} \binom{n}{k}$. A simple second moment computation shows that $|K_k(\Lambda)|$ is concentrated around its expectation, so with high probability for large n we have that $|K_k(\Lambda)| \geq \frac{1}{10} k^{-k} n^k$. Thus if $\mu_0 = \frac{1}{20} \frac{k-1}{k^{k+1}}$, we have that

$$0 = |H \cap K_k(\Lambda)| < \frac{k-1}{k} |K_k(\Lambda)| - \mu n^k.$$

\square

Lemma 11. Let $r = r_{k-1}(K_k^{(k-1)}, \dots, K_k^{(k-1)})$ be the k -color Ramsey number, where the $(k-1)$ -sets are colored and a monochromatic k -clique is forced. Then $A_n^{(k)}$ has no copy of $K_r^{(k)}$.

Proof. Let $X \subseteq V(A_n^{(k)})$ be such that $|X| = r$ and $A_n^{(k)}[X]$ is a clique. Then by the property of r , there exists a $Y \subseteq X$ such that $|Y| = k$ and all $(k-1)$ -subsets of Y have the same color c . But now

$$\sum_{\substack{T \subseteq Y \\ |T|=k-1}} f(T) = ck = 0 \pmod{k}.$$

Thus $Y \notin A_n^{(k)}$, which contradicts that $A_n^{(k)}[X]$ is a clique. \square

To show that $A_n^{(k)}$ satisfies $\text{Disc}^{(k)}(\mathcal{I}, \geq p, \mu)$ when $\mathcal{I} \neq \binom{[k]}{k-1}$, we will use a theorem of Towsner [38] that equates \mathcal{I} -discrepancy with counting \mathcal{I} -adapted hypergraphs. Therefore, we prove that the count of any \mathcal{I} -adapted hypergraph F in $A_n^{(k)}$ is correct with high probability.

Lemma 12. *Let $p = \frac{k-1}{k}$ and let $\mathcal{I} \subseteq 2^{[k]}$ be an antichain such that $\mathcal{I} \neq \binom{[k]}{k-1}$. Let F be an \mathcal{I} -adapted k -graph. For every $\mu > 0$, with probability going to one as n goes to infinity, the number of labeled copies of F in $A_n^{(k)}$ satisfies*

$$|\text{inj}[F \rightarrow A_n^{(k)}] - p^{|F|} n^{v(F)}| < \mu n^{v(F)}.$$

Proof. Let E_1, \dots, E_m be the ordering of edges in the definition of F being \mathcal{I} -adapted. First we show that if $Q : V(F) \rightarrow V(A_n^{(k)})$ is any injection, then the probability that $Q(E_i) \in A_n^{(k)}$ is exactly p independently of if the edges E_j with $j < i$ map to hyperedges or not. Indeed, since $\mathcal{I} \neq \binom{[k]}{k-1}$, let $I \in \binom{[k]}{k-1} - \mathcal{I}$. Now consider some E_i and let $\phi_i : E_i \rightarrow [k]$ be the bijection from the definition of F being \mathcal{I} -adapted. Now since $I \notin \mathcal{I}$, there is no $j < i$ such that $\phi_i(E_i \cap E_j) = I$. Thus conditioning on if the edges E_j with $j < i$ map to edges of $A_n^{(k)}$ or not potentially fixes the colors on $(k-1)$ -subsets of $Q(E_i)$ besides the $(k-1)$ -subset indexed by I . Since the color of $\{Q(x) : x \in E_i, \phi_i(x) \in I\}$ (which has size $k-1$) has probability exactly p to make the color sum of $Q(E_i)$ once all other colors are fixed, with probability p we have that $Q(E_i)$ is an edge.

Therefore, the probability that Q is an edge-preserving map is $p^{|F|}$. This implies that the expected number of labeled copies of F in $A_n^{(k)}$ is $p^{|F|} n(n-1) \cdots (n-v(F)+1)$. A simple second moment calculation shows that with high probability the number of labeled copies of F in $A_n^{(k)}$ is $p^{|F|} n^{v(F)} \pm \mu n^{v(F)}$ for large n . \square

Lastly, we need to show that $A_n^{(k)}$ satisfies $\text{Disc}^{(k-1)}(\mathcal{J}, \geq \alpha, \mu)$ in every link for every \mathcal{J} . We could do that similar to the previous lemma by showing that the count of \mathcal{J} -adapted k -graphs is correct, but instead are able to directly show that $\text{Disc}^{(k-1)}(\mathcal{J}, \geq \alpha, \mu)$ holds.

Lemma 13. *Let $\mathcal{J} \subseteq 2^{[k-1]}$ be an antichain and $\alpha = \frac{k-1}{k}$. Then for every $\mu > 0$, with probability going to one as n goes to infinity, $L(x)$ satisfies $\text{Disc}^{(k-1)}(\mathcal{J}, \geq \alpha, \mu)$ for each $x \in V(A_n^{(k)})$.*

Proof. Fix $x \in V(A_n^{(k)})$ and view $L_{A_n^{(k)}}(x)$ as a probability distribution over $(k-1)$ -graphs with vertex set $V(A_n^{(k)}) - x$. That is, an element from this probability distribution is generated by first generating $A_n^{(k)}$ and then outputting the link of x . We claim that the probability distribution $L(x)$ is isomorphic to the probability distribution $G^{(k-1)}(n-1, \alpha)$. To see this, consider $S \in \binom{V(A_n^{(k)}) - x}{k-1}$. Then $S \in L(x)$ if

$$\sum_{\substack{T \subseteq S \cup \{x\} \\ |T|=k-1}} f(T) \not\equiv 0 \pmod{k}.$$

We could rewrite this as

$$f(S) \not\equiv \sum_{\substack{T \subseteq S \\ |T|=k-2}} f(T \cup x) \pmod{k}.$$

The sum on the left hand side is some integer w_S between 0 and $k-1$, so that S is a hyperedge of $L(x)$ if and only if the color of S is not w_S . Since this is for every S and the colors assigned to S are mutually independent, $L(x)$ is isomorphic to $G^{(k-1)}(n-1, \alpha)$.

The proof is now complete, since for large n $G^{(k-1)}(n-1, \alpha)$ satisfies $\text{Disc}^{(k-1)}(\mathcal{J}, \geq \alpha, \mu)$ with very high probability as follows. Fix any \mathcal{J} -layout Λ . Each $(k-1)$ -clique in Λ is a hyperedge with probability α and two $(k-1)$ -cliques are independent unless one is a permutation of the other. So divide $K_{k-1}(\Lambda)$ up into at most $(k-1)!$ sets $R_1, \dots, R_{(k-1)!}$ such that within a single R_i there are no $(k-1)$ -tuples which are permutations of each other. Then the expected size of $H \cap R_i$ is $\alpha|R_i|$ and by Chernoff's inequality,

$$\mathbb{P}\left[\left||H \cap R_i| - \alpha|R_i|\right| > \epsilon n^{k-1}\right] < 2e^{-\epsilon^2 n^{2k-2}/2|R_i|}.$$

Since $|R_i| \leq n^{k-1}$, the probability is at most $e^{-cn^{k-1}}$ for some constant c . There are $(k-1)!$ sets R_i and there are at most $2^{k-2}2^{n^{k-2}}$ \mathcal{J} -layouts Λ , so with probability at most $e^{-c'n^{k-1}}$, the link of x fails $\text{Disc}^{(k-1)}(\mathcal{J}, \geq \alpha, \mu)$. There are n vertices of $A_n^{(k)}$, so with probability at most $ne^{-c'n^{k-1}} \rightarrow 0$, there is some vertex x of $A_n^{(k)}$ whose link fails $\text{Disc}^{(k-1)}(\mathcal{J}, \geq \alpha, \mu)$. \square

Proof of Proposition 3. As mentioned previously, to show that $A_n^{(k)}$ satisfies $\text{Disc}^{(k)}(\mathcal{I}, \geq p, \mu)$, we combine Lemma 12 with a theorem of Towsner [38] which is stated in the language of k -graph sequences. Converting from the probability distribution $A_n^{(k)}$ to a k -graph sequence is very similar to the proofs of [29, Lemmas 30 and 31] so we only briefly sketch the technique here. By the previous lemmas and the probabilistic method, for every $\mu > 0$ there exists an n_0 such that for every $n \geq n_0$ there exists some k -graph satisfying the properties in the previous lemmas (has the right edge density, fails $\text{Disc}^{(k)}(\binom{[k]}{k-1}, \geq p, \mu)$, no copy of K_r , has the right count of all \mathcal{I} -adapted hypergraphs, and satisfies $\text{Disc}^{(k-1)}(\mathcal{J}, \geq \alpha, \mu)$ in the links). Construct a k -graph sequence $\mathcal{H} = \{H_n\}_{n \in \mathbb{N}}$ by diagonalization by setting $\mu = \frac{1}{n}$.

By Lemma 12, \mathcal{H} satisfies the property that for every \mathcal{I} -adapted F , $\lim_{n \rightarrow \infty} t_F(H_n) = p^{|F|}$ so by [38, Theorem 1.1] \mathcal{H} is $\text{Disc}_p[\mathcal{I}]$ (where $t_F(H_n)$ and $\text{Disc}_p[\mathcal{I}]$ are defined in [38]). Thus for large n , the k -graphs in the sequence \mathcal{H} are the k -graphs which prove Proposition 3. \square

Proof of Proposition 4. Let $G = G^{(k)}(n, p)$ be the random k -graph with density p . Modify G by picking a single vertex $x \in V(G)$, removing all edges which contain x , and adding edges so that $L(x) = A_n^{(k-1)}$. Now the link of x has no copy of $K_r^{(k-1)}$ so that G has no perfect $K_{r+1}^{(k)}$ -packing. Also, G satisfies $\text{Disc}^{(k)}(\binom{[k]}{k-1}, \geq p, \mu)$ since the random k -graph satisfies $\text{Disc}^{(k)}(\binom{[k]}{k-1}, \geq p, \mu)$ (see the proof of Lemma 13) and we only modified at most n^{k-1} hyperedges. By the previous lemmas, the link of x fails $\text{Disc}^{(k-1)}(\binom{[k-1]}{k-2}, \geq \alpha, \mu)$ and satisfies $\text{Disc}^{(k-1)}(\mathcal{J}, \geq \alpha, \mu)$ for all $\mathcal{J} \neq \binom{[k-1]}{k-2}$. \square

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References

- [1] N. Alon and R. Yuster. H -factors in dense graphs. *J. Combin. Theory Ser. B*, 66(2):269–282, 1996.
- [2] F. Chung. Quasi-random hypergraphs revisited. *Random Structures Algorithms*, 40(1):39–48, 2012.
- [3] F. R. K. Chung. Quasi-random classes of hypergraphs. *Random Structures Algorithms*, 1(4):363–382, 1990.
- [4] F. R. K. Chung and R. L. Graham. Quasi-random hypergraphs. *Random Structures Algorithms*, 1(1):105–124, 1990.
- [5] F. R. K. Chung and R. L. Graham. Quasi-random set systems. *J. Amer. Math. Soc.*, 4(1):151–196, 1991.
- [6] F. R. K. Chung and R. L. Graham. Cohomological aspects of hypergraphs. *Trans. Amer. Math. Soc.*, 334(1):365–388, 1992.
- [7] F. R. K. Chung, R. L. Graham, and R. M. Wilson. Quasi-random graphs. *Combinatorica*, 9(4):345–362, 1989.
- [8] D. Conlon, H. Hàn, Y. Person, and M. Schacht. Weak quasi-randomness for uniform hypergraphs. *Random Structures Algorithms*, 40(1):1–38, 2012.
- [9] A. Czygrinow, L. DeBiasio, and B. Nagle. Tiling 3-uniform hypergraphs with $k_4^3 - 2e$. to appear in *Journal of Graph Theory*.
- [10] D. Dellamonica, Jr. and V. Rödl. Hereditary quasirandom properties of hypergraphs. *Combinatorica*, 31(2):165–182, 2011.
- [11] A. Hajnal and E. Szemerédi. Proof of a conjecture of P. Erdős. In *Combinatorial theory and its applications, II (Proc. Colloq., Balatonfüred, 1969)*, pages 601–623. North-Holland, Amsterdam, 1970.

- [12] H. Hàn, Y. Person, and M. Schacht. On perfect matchings in uniform hypergraphs with large minimum vertex degree. *SIAM J. Discrete Math.*, 23(2):732–748, 2009.
- [13] P. Keevash. The existence of designs. <http://arxiv.org/abs/1401.3665>.
- [14] P. Keevash. A hypergraph blow-up lemma. *Random Structures Algorithms*, 39(3):275–376, 2011.
- [15] P. Keevash and R. Mycroft. A geometric theory for hypergraph matching. to appear in Mem. Amer. Math. Soc.
- [16] I. Khan. Perfect matchings in 4-uniform hypergraphs. arXiv:1101.5675.
- [17] I. Khan. Perfect matchings in 3-uniform hypergraphs with large vertex degree. *SIAM J. Discrete Math.*, 27(2):1021–1039, 2013.
- [18] Y. Kohayakawa, B. Nagle, V. Rödl, and M. Schacht. Weak hypergraph regularity and linear hypergraphs. *J. Combin. Theory Ser. B*, 100(2):151–160, 2010.
- [19] Y. Kohayakawa, V. Rödl, and J. Skokan. Hypergraphs, quasi-randomness, and conditions for regularity. *J. Combin. Theory Ser. A*, 97(2):307–352, 2002.
- [20] J. Komlós, G. N. Sárközy, and E. Szemerédi. Blow-up lemma. *Combinatorica*, 17(1):109–123, 1997.
- [21] J. Komlós, G. N. Sárközy, and E. Szemerédi. Proof of the Alon-Yuster conjecture. *Discrete Math.*, 235(1-3):255–269, 2001. Combinatorics (Prague, 1998).
- [22] D. Kühn and D. Osthus. Loose Hamilton cycles in 3-uniform hypergraphs of high minimum degree. *J. Combin. Theory Ser. B*, 96(6):767–821, 2006.
- [23] D. Kühn and D. Osthus. The minimum degree threshold for perfect graph packings. *Combinatorica*, 29(1):65–107, 2009.
- [24] D. Kühn, D. Osthus, and T. Townsend. Fractional and integer matchings in uniform hypergraphs. <http://arxiv.org/abs/1304.6901>.
- [25] D. Kühn, D. Osthus, and A. Treglown. Matchings in 3-uniform hypergraphs. *J. Combin. Theory Ser. B*, 103(2):291–305, 2013.
- [26] J. Lenz and D. Mubayi. Eigenvalues and linear quasirandom hypergraphs. submitted. <http://arxiv.org/abs/1208.4863>.
- [27] J. Lenz and D. Mubayi. Eigenvalues of non-regular linear quasirandom hypergraphs. online at <http://arxiv.org/abs/1309.3584>.
- [28] J. Lenz and D. Mubayi. Perfect packings in quasirandom hypergraphs. online at <http://arxiv.org/abs/1402.0884>.

- [29] J. Lenz and D. Mubayi. The poset of hypergraph quasirandomness. accepted in Random Structures and Algorithms. <http://arxiv.org/abs/1208.5978>.
- [30] A. Lo and K. Markström. F-factors in hypergraphs via absorption. preprint arXiv:1105.3411.
- [31] A. Lo and K. Markström. Minimum codegree threshold for $(K_4^3 - e)$ -factors. *J. Combin. Theory Ser. A*, 120(3):708–721, 2013.
- [32] K. Markström and A. Ruciński. Perfect matchings (and Hamilton cycles) in hypergraphs with large degrees. *European J. Combin.*, 32(5):677–687, 2011.
- [33] O. Pikhurko. Perfect matchings and K_4^3 -tilings in hypergraphs of large codegree. *Graphs Combin.*, 24(4):391–404, 2008.
- [34] V. Rödl, A. Ruciński, and E. Szemerédi. Perfect matchings in large uniform hypergraphs with large minimum collective degree. *J. Combin. Theory Ser. A*, 116(3):613–636, 2009.
- [35] A. Shapira¹ and R. Yuster. The quasi-randomness of hypergraph cut properties. *Random Structures Algorithms*, 40(1):105–131, 2012.
- [36] A. Thomason. Pseudorandom graphs. In *Random graphs '85 (Poznań, 1985)*, volume 144 of *North-Holland Math. Stud.*, pages 307–331. North-Holland, Amsterdam, 1987.
- [37] A. Thomason. Random graphs, strongly regular graphs and pseudorandom graphs. In *Surveys in combinatorics 1987 (New Cross, 1987)*, volume 123 of *London Math. Soc. Lecture Note Ser.*, pages 173–195. Cambridge Univ. Press, Cambridge, 1987.
- [38] H. Towsner. Sigma-algebras for quasirandom hypergraphs. available online at <http://arxiv.org/abs/1312.4882>.
- [39] A. Treglown and Y. Zhao. Exact minimum degree thresholds for perfect matchings in uniform hypergraphs. *J. Combin. Theory Ser. A*, 119(7):1500–1522, 2012.
- [40] A. Treglown and Y. Zhao. Exact minimum degree thresholds for perfect matchings in uniform hypergraphs II. *J. Combin. Theory Ser. A*, 120(7):1463–1482, 2013.